Backstepping Design of Composition Cascade Control for Distillation Columns

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Although the cascade control of distillation columns had been extensively studied, some aspects still remain unexplored or unresolved. For instance, a unifying approach is needed to systematize the existing ad hoc controller constructions, to rigorously explain their remarkably robust property in using rather simple input-output linear models, and to explore the possibility of improving their construction and functioning. Some of these problems are addressed by the recently developed backstepping-passivation approach in constructive nonlinear control, yielding a systematic controller construction coupled to a simple tuning scheme that can be executed with standard tuning rules, a closed-loop stability criterion, an explanation of the closed-loop dynamics behavior, and controllers that perform better than existing ones. The applicability of the theoretical results and their controller construction-tuning schemes are illustrated by considering the single- and dual-point control cases of a binary acetone-ethanol distillation column.

Introduction

The main task in the operation and control of distillation columns is the regulation of the products (distillate and residual) composition despite load disturbances. In practice, composition regulation is approached via indirect methodologies where a specified tray temperature is regulated at a given set point, which, in principle, corresponds to the desired composition value. In general, the selection of the temperature set point is left for manual control by the operators. Distillation columns are often operated in this manner due to slow (delayed) or lacking composition measurements. However, this approach has the problem that unavoidable model uncertainties make tight composition control very difficult. For instance, load disturbances (such as feed flow, temperature, and composition) may unmatch the prescribed (steady-state) temperature corresponding to the composition set point, which in turn induces a steady-state offset in composition. In practice, this problem converts the manual control by the operator into a trial-and-error one to set the "correct" temperature set point. A possible solution to this problem is to close the loop directly from composition measurements. However, there is often a long time-delay associated with measuring the product compositions, which makes fast control impossible (Wolff and Skogestad, 1996) and yields poor control performance against load disturbances.

From an industrial practice viewpoint, it would be desirable to endow the controllers with servo responses (zero steady-state error in composition) and load responses to protect the operation against external disturbances (such as in feed flow and reboiler heating conditions). When fast temperatures and slow (delayed) composition measurements are available, a common approach to address the problem is cascade (also called indirect feedforward, Shen and Yu (1992)) control designs. Basically, cascade control is a linear two-loop feedback that employs both an external composition loop for servo responses, and an inner temperature loop for load damping purposes. The end results are: (1) a faster transient response induced by the temperature loop; (2) an easier-to-design control loop; and (3) faster overload protection. The rationale behind the good disturbance rejection capability of cascade control is that the feed upsets will affect the tray

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temperature faster than the product composition, and thus compensation for the upset can begin sooner than if (delayed) composition control alone is used (McAvoy et al., 1996).

The composition control of distillation columns has been the object of recent research (see, for instance, Krishnaswamey et al. (1990) and Shen and Yu (1992)). These results mainly recommended using cascades when the delay in the secondary loop is much smaller than that in the primary loop. Brambilla et al. (1994) discussed an approach to the design of parallel cascade controllers for multicomponent distillation columns. Specifically, they studied the selection of cascade loops and the use of a nonlinear filter between the cascade loops. Wolff and Skogestad (1996) addressed the case of distillation columns. Based on frequency analysis tools, they focused mainly on the quantification of improvements in loop interaction and disturbance rejection induced by adding a secondary temperature loop. McAvoy et al. (1996) proposed an inferential parallel cascade control to improve the performance of plant-wide control systems. The approach was tested on a benchmark problem (Tennessee Eastman process), yielding an important reduction of the composition variability. However, stability aspects of the proposed approach were not studied. Suitably tuned PID controllers were used in both the primary (composition) and secondary (temperature) loops, which provided acceptable disturbance rejection and servo responses.

Despite the successful functioning of such composition cascade control designs for distillation columns, there are still some issues that deserve further study (McAvoy et al., 1996; Wolff and Skogestad, 1996). For instance, a unifying approach is needed to systematize the existing *ad hoc* controller constructions to rigorously explain their remarkable robustness property with respect to the employment of rather simple input-output linear models, and to explore the possibility of improving their construction and functioning. In the present work, some aspects of these problems are addressed by resorting to the recently developed backstepping-passivation approach in constructive control (Sepulchre et al., 1997), yielding a systematic controller construction coupled to a simple tuning scheme that can be executed with standard tuning rules, a closed-loop stability criterion, an explanation of the closed-loop dynamics behavior, and controllers that perform better than the existing ones. The applicability of the theoretical results and their controller construction-tuning schemes are illustrated by considering the single and dual-point control cases of a binary acetone-ethanol distillation column.

With respect to existing results in the literature (see, for instance, Brambilla et al. (1994), Wolff and Skogestad (1996) and McAvoy et al., 1996), the contribution of the present work can be summarized as follows:

(a) The nature of composition cascade control design is clarified in terms of backstepping designs. This allows a systematic cascade control design procedure with easy-to-implement controllers.

(b) With a framework that is provided by backstepping design procedures, the stability and performance of the underlying closed-loop system is guaranteed.

(c) Systematic extensions for multicascade control configuration are obtained as a natural consequence of the backstepping design procedure.

The results in this article should be seen as a step-ahead to previous results in cascade control of distillation columns (Brambilla et al., 1994; Wolff and Skogestad, 1996; McAvoy et al., 1996). In particular, the time-domain backstepping procedure taken in this article provides further insight on the functioning and improvement of composition cascade control designs.

Cascade Control Problem

Consider the binary distillation column presented in Figure 1, where x_D (or x_B) is the distillate (or bottom) volatile component composition to be regulated, R is the reflux ratio, and V is the vapor flow rate steaming from the reboiler, and T_m (or T_n) is the temperature at some tray in the rectifying (or stripping) section. In the single-point control case, R (or V) is the manipulated variable, x_D (or x_B) is the controlled or primary measured output, and T_m (or T_n) is the secondary temperature measurement. For the sake of simplicity, we will restrict ourselves to the single-point control problem in the rectifying section where the distillate composition x_D is the controlled variable, and the reflux flow rate R is the manipulated variable, in the understanding that the proposed control design approach can be applied and the stability analysis can be extended along the same steps to dual composition control.

Assume that the temperature T_m is measured without time-delays, and the distillate composition x_D is measured or inferred with time-delay $\theta \geq 0$. In this way, x_D and T_m are seen, respectively, as the primary and the secondary measurements. A distillation column is an open-loop stable process for most operating conditions. In this case, a feedback control scheme is expected to have better servo and disturbance rejection properties than an open-loop operation.

As it is done in most distillation control studies (see, for instance, Morari and Zafiriou, 1989), numerical simulations will be carried out on a detailed binary mixture model. Figure 2 presents a typical step response of a binary (acetone-

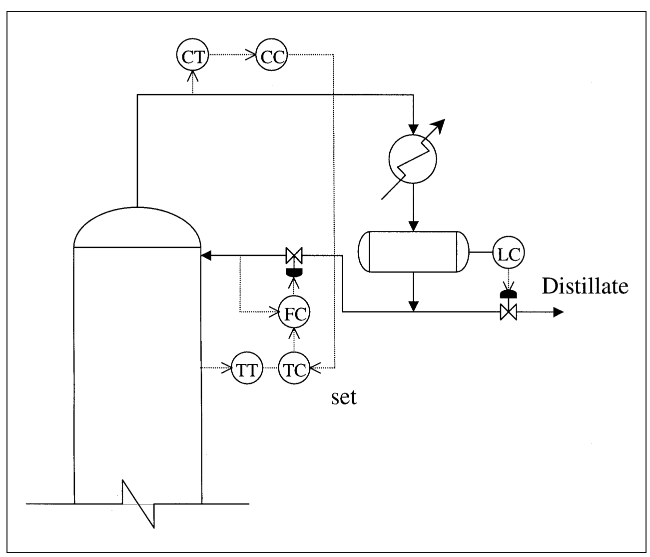

Figure 1. Distillation column and temperature cascade control configuration to regulate the distillate composition.

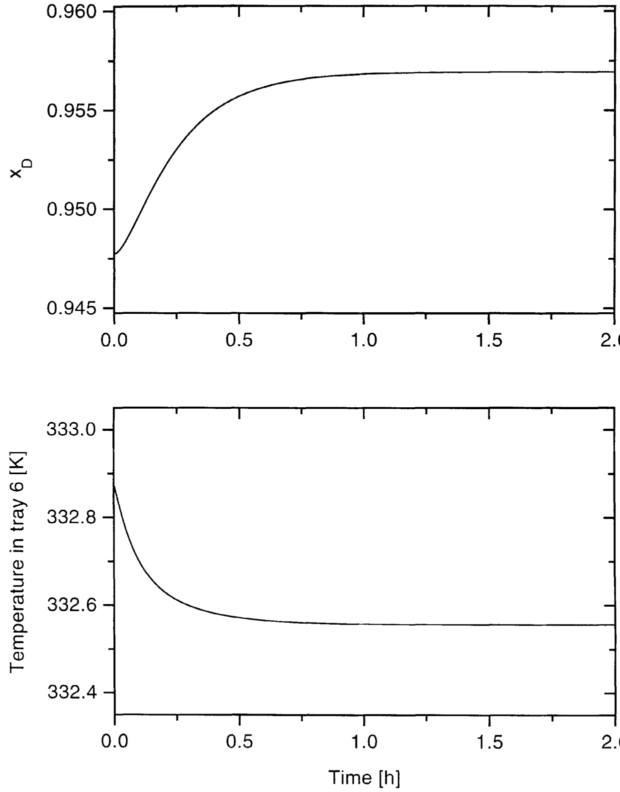

Figure 2. Open-loop response under a+5% step change in the reflux flow rate.

ethanol) distillation column model where the measured temperature is in tray 6. Such a response is smooth, almost-monotonous, and convergent. Consequently, it is reasonable to model the input-output responses with the simple models (Chien et al., 1997)

$$\frac{\Delta x_D}{\Delta R} = \frac{K_{Rx}}{\tau_{Rx} s + 1} \tag{1}$$

and

$$\frac{\Delta T_m}{\Delta R} = \frac{K_{RT}}{\tau_{RT} s + 1} \tag{2}$$

where $s = d/dt$ is the time-derivative operator, $\tau_{Rx} > 0$ and $\tau_{RT} > 0$ are the dominant time constants; K_{Rx} and K_{RT} are steady-state gains, $\Delta R = R - \overline{R}$, $\Delta x_D = x_D - \bar{x}_D$, $\Delta T_m = T_m - \overline{T}_m$; and $\overline{R}$, $\bar{x}_D$ and $\overline{T}_m$ are, respectively, nominal reflux flow rate, distillate composition, and measured tray temperature. Since x_D is subjected to a measurement delay θ, the model (Eq. 1) can be completed as follows

$$\frac{\Delta x_D}{\Delta R} = \frac{K_{Rx} \exp(-\theta s)}{\tau_{Rx} s + 1} \tag{3}$$

Equations 2 and 3 become empirical models of the distillation dynamics around the nominal operating point ($\bar{x}_D$, $\overline{T}_m$, $\overline{R}$).

Remark 1. In practice, a time-delay $\theta_T > 0$ induced by tray flow dynamics is always present in the input-output path $\Delta R \rightarrow \Delta T_m$. In this article, we will assume that such time delay can be neglected as compared to that in the input-output path $\Delta R \rightarrow \Delta x_D$. In fact, if θ_T is of the order or much larger than θ, no improvements in composition control can be expected from the incorporation of a secondary temperature loop (Krishnaswamey et al., 1990).

Remark 2. Although Eqs. 2 and 3 are simple one time-constant models, they retain the main characteristics of composition dynamics (see, for instance, Morari and Zafiriou, 1989). Comments on the use of higher-order models in the design of composition cascade control and stability of the underlying closed-loop system will be made in the section on stability analysis.

The basic control problem is to maintain the distillate composition at a desired value $x_{D,\mathrm{ref}}$ (contained in a neighborhood of the nominal distillate composition $\bar{x}_D$) by manipulating the reflux flow rate R. To this end, a feedback controller must be designed based on the models (Eq. 2 and Eq. 3).

Since the temperature measurements are inexpensive and with no time-delay, the resulting controller should make use of both composition $\exp(-\theta s) x_D$ and temperature T_m measurements to yield servo responses (set point tracking) and disturbance rejection. Since the primary measurement $\exp(-\theta s) x_D$ is subjected to a large time-delay and the secondary measurement, T_m is obtained with no delay; a cascade control structure should be employed to achieve the control objectives (Krishnaswamey et al., 1990; Krishnaswamey and Ragaiah, 1992). In particular, in this article we are interested in finding a means to systematize the existing construction-tuning procedures, in finding out whether it is possible to obtain control design with enhanced performance and robustness properties, and in obtaining a construction-tuning procedure coupled with a robust stability condition.

Cascade Control Construction

In this section, the composition-temperature distillation cascade controller is built on the basis of the rationale associated to the backstepping approach in constructive robust control (Sepulchre et al., 1997). The purpose is twofold: to obtain a controller construction that unifies and systematizes the existing *ad hoc* constructions, and to open the possibility of endowing the control design with a robust stability analysis for the resulting plant-controller closed-loop system.

Essentially, the backstepping approach is a structure oriented design procedure to achieve a high-performance vs. robustness compromise via passivation, or, equivalently, by considering measured states in the input-output path as virtual controls in order to approximate a (controllable-observable) high-order (that is, relative degree) control model by a set of low-order (usually linear) models in cascade interconnection in such a way that the modeling error estimation and compensation capabilities inherent to the virtual control (that is, secondary) measurements are exploited to endow the control design with a high degree of robustness to modeling errors at a minimum cost in performance compared with the one attained by the underlying (possible nonlinear and with poor robustness) high-order controller without modeling errors and driven by its measured high-dimensional full state. The methodological rationale is rigorously developed by the structure-stability oriented constructive robust control (Sepulchre et al., 1997), and is the one that can rigorously explain the remarkable robustness property of the well-known and accepted cascade controllers in the chemical engineering community in general and in the column distillation field in particular. These considerations suggest to us that the backstepping-passivation approach and the robust stability oriented constructive nonlinear method constitute the appropriate tools to unify and systematize the existing *ad hoc* control constructions, to rigorously assess their capabilities and limitations, and to improving their functioning.

Master controller

Following the backstepping approach, let us regard the tray temperature T_m as a "virtual" control input to regulate the distillate composition x_D about its prescribed reference value $x_{D,\mathrm{ref}}$. For this purpose, combine Eqs. 2 and 3 and obtain the input (T_m)-output (x_D) model

$$\frac{\Delta x_D}{\Delta T_m} = K_{Tx}\left(\frac{\tau_{RT}s+1}{\tau_{Rx}s+1}\right)\exp(-\theta s),\, K_{Tx} = K_{Rx}/K_{RT} \quad (4)$$

If $\Delta T_{m,\mathrm{ref}} = T_{m,\mathrm{ref}} - \overline{T}$ is used instead the temperature deviation ΔT_m, the model to compute the master controller is

$$\frac{\Delta x_D}{\Delta T_{m,\mathrm{ref}}} = G_M(s) \stackrel{\mathrm{def}}{=} K_{Tx}\left(\frac{\tau_{RT}s+1}{\tau_{Rx}s+1}\right)\exp(-\theta s) \quad (5)$$

Since the dynamics have a relatively large time-delay that limits the achievable bandwidth (Morari and Zafiriou, 1989), McAvoy et al. (1996) have suggested to add dynamic compensation to improve the response of the primary loop, and this was confirmed by Bonnet et al. (2000) by showing that the parameterization of all stabilizing controllers is a family of low-gain type. From these considerations, it follows that, if the time-constant τ_{Rx} of the input (R)-output (x_D) model (Eq. 5) is sufficiently small to give an acceptable convergence rate, a low-gain pure integral controller

$$\frac{\Delta T_{m,\mathrm{ref}}}{\Delta x_D} = \frac{1}{K_{Tx}\tau_{M,I}s} \quad (6)$$

with $\tau_{M,I} > 0$ being the master integral time-constant, suffices to regulate the distillate composition. In time-domain, the representation of the controller (Eq. 6) is given by

$$T_{m,\mathrm{ref}} = \overline{T}_m + K_{Tx}^{-1}\tau_{M,I}^{-1}\int_0^t (x_{D,\mathrm{ref}} - x_D(\sigma))\,d\sigma \quad (7)$$

Remark 3. The low-gain master controller (Eq. 7) is robust against parametric and input-related non-modeled dynamics (Bonnet et al., 2000), in the sense that it stabilizes the actual plant

$$\frac{\Delta x_D}{\Delta T_m} = K_{Tx}\left(\frac{\tau_{RT}s+1}{\tau_{Rx}s+1}\right)\exp(-\theta s)[1+\Delta(s)]$$

with an additive (unstructured) uncertainty $\Delta(s)$.

Slave controller

Having built the master controller to regulate the distillate composition by manipulating the trajectory $T_{m,\mathrm{ref}}$ of the measured tray temperature, the task for the slave control design is to obtain a controller that manipulates the reflux flow rate R such that the tray temperature tracks the time-varying set point $T_{m,\mathrm{ref}}(t)$ generated by the master controller.

Let us recall the input (R)-output (T_m) model (Eq. 2) and rewrite it in time-domain form ($\dot{T}_m$ stands for the time-derivative dT_m/dt)

$$\dot{T}_m = \tau_{RT}^{-1}(-T_m + \overline{T}_m) + \tilde{K}_{RT}^{-1}(R - \overline{R}) + \eta,\ \tilde{K}_{RT} = \tau_{RT}^{-1}K_{RT} \quad (8)$$

with an important modification: the unknown but observable term η has been added so that the preceding model exactly matches the input (R)-output (T_m) column dynamics. In other words, η accounts for the model errors associated to the model (Eq. 2). For the moment, assume that the input-output modeling error term η is known. Require the closed-loop temperature tracking error $\xi_m \stackrel{\mathrm{def}}{=} T_m - T_{m,\mathrm{ref}}$ dynamics to be

$$\dot{\xi}_m = -\tau_{S,c}^{-1}\xi_m \quad (9)$$

with adjustable closed-loop time-constant $\tau_{S,c} > 0$. Combine Eq. 9 with Eq. 8 and obtain the "exact" slave controller

$$R^{\mathrm{ex}} = \varphi_S^{\mathrm{ex}}(x_D, T_m, \eta) \stackrel{\mathrm{def}}{=} \overline{R} + \tilde{K}_{RT}^{-1}\left[\dot{T}_{m,\mathrm{ref}} + \tau_{RT}^{-1}(T_{m,\mathrm{ref}} - \overline{T}_m) - (\tau_{c,S}^{-1} - \tau_{RT}^{-1})\xi_m - \eta\right] \quad (10)$$

Since the matching term η is timewise uniquely determined from the input ΔR, the output ΔT_m, and its time-derivative, the term η is locally weakly observable (Hermann and Krener, 1977; Diop and Fliess, 1991), and, thus, in turn implies that an arbitrarily fast online reconstruction $\hat{\eta}$ of η can be done via a dynamic observer. In particular, η can be reconstructed with the reduced-order observer

$$\dot{\hat{\eta}} = \tau_{S,e}^{-1}\left[\dot{T}_m - \tau_{RT}^{-1}(-T_m + \overline{T}_m) - \tilde{K}_{RT}^{-1}(R - \overline{R}) - \hat{\eta}\right] \quad (11)$$

where $\tau_{S,e} > 0$ is the observer time-constant. To circumvent the need of the time-derivative $\dot{T}_m$, let us rewrite the preceding observer in the proper form

$$\begin{aligned}\dot{w} &= -\tau_{RT}^{-1}(-T_m + \overline{T}_m) - \tilde{K}_{RT}^{-1}(R - \overline{R}) - \hat{\eta},\\ w(0) &= -T_m(0)\\ \hat{\eta} &= \tau_{S,e}^{-1}(w + T_m)\end{aligned} \quad (12)$$

The combination of this observer with the saturated version of the "exact" slave controller (with η replace by its estimate $\hat{\eta}$) yields its observer-based approximation, or, equivalently, our candidate slave measurement-driven controller

$$\dot{w} = -\tau_{RT}^{-1}\left(-T_m + \overline{T}_m\right) - \tilde{K}_{RT}^{-1}(R - \overline{R}) - \tau_{S,e}^{-1}(w + T_m),$$
$$w(0) = -T(0)$$
$$R^c = \varphi_S^{\text{ex}}(x_{D,m}, T_m, \hat{\eta})$$
$$R = \text{Sat}(R^c;\, R_{\min},\, R_{\max}) \tag{13}$$

where R^c is the computed control input and *Sat* is the saturation function

$$R = \text{Sat}(R^c;\, R_{\min},\, R_{\max}) \overset{\text{def}}{=} \begin{cases} R_{\min} & \text{if} \quad R^c \le R_{\min} \\ R^c & \text{if} \quad R_{\min} < R^c < R_{\max} \\ R_{\max} & \text{if} \quad R^c \ge R_{\max} \end{cases} \tag{14}$$

which is introduced to meet the physical constraint $R \in [R_{\min}, R_{\max}]$.

In summary, the application of a backstepping procedure in conjunction with a modeling error estimator led us to a candidate cascade control configuration to control the distillate composition. Observe that, if $\hat{\eta}$ tends fastly to η, the preceding observer-based controller tends to the one built with the "exact" slave controller (driven by the actual η). As we shall see later, comparing with earlier cascade control designs, this capability of fastly reconstructing the modeling error η in the input (R)-output (T_m) model signifies an enhanced performance-robustness capability of the proposed controller.

Remark 4. After some straightforward algebraical manipulations, the computed slave controller (Eq. 13) can be expressed as a linear PI-type controller with antireset windup acting on the manipulated variable R

$$R^c = \overline{R} + \overbrace{K_{P,T}(T_{m,\text{ref}} - T_m) + K_{I,T}\int_0^t \left(T_{m,\text{ref}}(\sigma) - T_m(\sigma)\right) d\sigma}^{\text{Temperature } PI \text{ feedback}}$$
$$+ \overbrace{\tilde{K}_{RT}^{-1}\dot{T}_{m,\text{ref}}}^{\text{master feedforward term}} + \overbrace{\tau_{S,e}^{-1}\int_0^t [R(\sigma) - R^c(\sigma)]\, d\sigma}^{\text{Antireset windup}} \tag{15}$$

where the temperature gains are given by

$$K_{P,T} = \tilde{K}_{RT}^{-1}\left(-\tau_{RT}^{-1} + \tau_{S,c}^{-1} + \tau_{S,e}^{-1}\right)$$
$$K_{I,T} = \tilde{K}_{RT}^{-1}\tau_{S,c}^{-1}\tau_{S,e}^{-1} \tag{16}$$

Notice that the temperature feedback is basically a PI controller acting on the tracking error $\xi(t)$ with a novel gain parameterization given by (Eq. 16) in terms of the desired closed slave-loop time-constant $\tau_{S,c}$ (see Eq. 9) and of the prescribed modeling error estimation time-constant $\tau_{S,e}$. The slave controller is endowed with the antireset windup scheme

$$\tau_{S,e}^{-1}\int_0^t [R(\sigma) - R^c(\sigma)]\, d\sigma \tag{17}$$

to compensate for control input saturations. This term compensates for deviates of the actual input $R(t)$ from the computed $R^c(t)$ one. In this way, when $R - R^c \neq 0$, the feedback signal $\tau_{S,e}^{-1}\int_0^t [R(\sigma) - R^c(\sigma)]d\sigma$ recomputes the integral action such that the modeling error estimator (Eq. 11) uses the right slave control input value R, preventing the slave controller from winding up (Kothare et al., 1994).

Thus, we conclude that the proposed master/slave feedback controller is equivalent to a parallel cascade controller with a low-gain composition feedback and a high-gain composition feedback. This control configuration presents the following features:

(a) The uncertainty η of the input-output $R \to T_m$ path is quickly estimated and compensated in the slave loop, and uncertainties and time-delay related to the input-output $T_m \to x_D$ path are compensated in the master loop via pure integral action. In this way, feed and reboiler disturbances are damped by virtue of the fast control action of the slave loop.

(b) In comparison with a traditional parallel cascade PI control where the temperature feedback uses a nominal temperature $\overline{T}_m$ to compute a feedback error $\overline{T}_m - T_m$, the cascade PI controller (Eq. 15) design uses the output of the master controller $T_{m,\text{ref}}$ to compute the tracking error $\xi(t) = T_{m,\text{ref}} - T_m$.

(c) If the modeling error signal $\eta(t)$ is estimated and cancelled out very quickly by choosing a sufficiently small estimation parameter $\tau_e > 0$, then T_m will converge quickly to $T_{m,\text{ref}}$. In this way, the performance induced by the master controller will be achieved as $T_m \to T_{m,\text{ref}}$. This leads to the conclusion that the overall performance is imposed by the master controller action.

(d) The proposed master/slave controller is protected against a composition sensor failure. In this case, the controller is simply a standard temperature regulator with constant reference $T_{m,\text{ref}}$.

Stability Analysis

Although cascade control structures are very popular in industrial applications, there is a lack of rigorous stability results to back up its functioning. In principle, the stability analysis would provide insight on control performance and would lead to improved cascade control designs (Yu and Luyben, 1985). The backstepping design procedure provides a natural framework to study the stability properties of the controlled distillation column. The idea for stability analysis is the following: (a) prove that the master controller with virtual control input $\Delta T_{m,\text{ref}}$ stabilizes the input-output path $\Delta T_{m,\text{ref}} \to \Delta x_D$; (b) since $\Delta T_{m,\text{ref}}$ is not a control input, prove that the tracking error ξ_m is bounded and vanishing; and (c) use singular perturbation tools to prove that the actual slave controller (Eq. 13) gives the desired properties on ξ_m.

To simplify presentation, and since the stability results are based on linear models of the actual distillation process, and the results should be of local nature, it will be considered that the controller output R^c is not subjected to saturations (that is, $R = R^c$). The main stability and performance result can be summarized as follows (a sketch of the proof is given in the Appendix).

Theorem 1. Consider the distillation dynamics (Eqs. 1–3). Assume that $\tilde{K}_{RT}^{-1}(\partial\eta/\partial R) + 1 > 0$. The following statements hold for the sufficiently small master integral time-constant $\tau_{M,I} > 0$ and the sufficiently large slave estimation time-constant $\tau_{S,e} > 0$:

(a) *(Robust stability)* The cascade control design as described in the previous section ensures regulation of the distillate composition in the sense that $x_D \to x_{D,\text{ref}}$ asymptotically.

(b) *(Robust performance)* Let $\xi(t,\xi_0;\tau_{S,e})$ be the trajectory of the tracking error with initial condition ξ_0 and estimation time-constant $\tau_{S,e} > 0$. Then $(\xi(t,\xi_0;\tau_{S,e}) \to \xi_0 \exp(-t/\tau_{S,c})$ as $\tau_{S,e} \to 0$. That is, the performance imposed by the exact slave controller (Eq. 10) is achieved as the estimation time-constant $\tau_{S,e}$ goes to zero.

The condition $\tilde{K}_{RT}^{-1}\,(\partial\eta/\partial R)+1>0$ establishes how accurate the high-frequency gain $\tilde{K}_{RT}$ must be estimated. In fact, Proposition A.3 (see the Appendix) implies that the slave feedback controller yields zero tracking error if the process high-frequency gain uncertainty satisfies $\tilde{K}_{RT}^{-1}\,(\partial\eta/\partial R)+1>0$, that is, if the gain error does not exceed 100%, or equivalently, if the gain does not change sign. When the gain changes sign a positive feedback loop results (see the Appendix) and the system becomes unstable. It should be stressed that the stability analysis for dual-point cascade control can be done with a suitable extension of the technique employed in the single point case.

Remark 5. Notice that, for large values of $\tau_{S,e}$, the parameterization (Eq. 16) defines a high-gain PI controller. In this way, asymptotic regulation of distillate composition is achieved via a low-gain master controller to avoid unstabilizing effects due to time-delays, in cascade configuration with a high-gain slave controller to ensure temperature tracking. Since temperature measurement delays can be neglected as compared to composition measurement delays, Theorem 1b implies that the major performance limitations are due to large time-delays in the master loop (see Ko and Edgar (2000) for a wide discussion on performance assessment of cascade control loops).

Remark 6. Simple input-output models as the ones used above are commonly used in the distillation control literature (see, for instance, Morari and Zafiriou, 1989; Wolff and Skogestad, 1996). Higher-order dynamical models as

$$\frac{\Delta x_D}{\Delta R} = G_{Rx}(s)\exp(-\theta s)$$

and

$$\frac{\Delta T_m}{\Delta R} = G_{RT}(s),$$

where $G_{Rx}(s)$ and $G_{RT}(s)$ are minimum-phase, stable transfer functions with $G_{Tx}(s) = G_{Rx}(s)/G_{RT}(s)$ being a rational transfer function, can be also used to design a temperature cascade controller. From the results reported by Bonnet et al. (2000), a low-order compensator with integral action $C_M(s)$ can be used as master controller. The slave controller is obtained from a signal tracking problem based on this input-output model. Following the modeling error compensation approach as described in Alvarez-Ramirez (1999), it results in a ρ_{RT}-order high-gain slave controller equipped with an integral action, where $\rho_{RT} \geq 1$ is the relative degree of $G_{Tx}(s)$. The stability analysis is more involved than that described in this article; nevertheless, it can be carried along the same steps.

As stated in the section on the statement of the problem, the proposed controller should be amenable to tuning with well-known rules, like the ones employed in industrial practice. In fact, this is the case for the proposed controller, and the corresponding rules are recalled and listed next:

(1) In a first step, choose the integral time-constant $\tau_{M,I} > 0$ so as to obtain closed-loop stability in the input-output $\Delta T_m \to \Delta x_D$ path. Following IMC ideas (see Chien and Fruehauf, 1990), choose $\tau_{M,I}$ not smaller than the underlying time-delay $\theta \geq 0$.

(2) The parameter $\tau_{S,e} > 0$ defines the rate of convergence of the estimation error. Since the slave (inner) loop is designed to track the temperature reference trajectory provided by the master (outer) loop, the slave loop should be faster than the master loop. In principle, the slave estimation time-constant $\tau_{S,e}$ should be chosen about 5 to 10 times smaller than the nominal closed-loop time-constant $\tau_{S,c}$. The rationale behind this recommendation is to estimate modeling error signals faster than the expected time-constant of the process. The tuning of the cascade PI controller structure is particularly easy to carry out in view of the fact that, up to the point where the influence of unmodeled dynamics and measurement noise is no longer negligible, the robustness of the closed-loop system increases monotonically with $\tau_{S,e}^{-1}$.

Extension to Multiple Temperature Measurements

The addition of a tray temperature as a secondary measurement enhances the disturbance rejection capabilities of distillate composition control. Since temperature measurements are fast and inexpensive, it is reliable to add more than one temperature measurement. By measuring temperature at several rectifying trays, disturbances are detected quickly, so as fast feedforward action can be obtained. The systematic procedure described before can be easily extended to the control design with multiple temperature measurements.

A brief outline of cascade composition control with multiple temperature measurements is given as follows. Let $(T_{m,1}, T_{m,2}, \ldots, T_{m,p})^T \in \mathbb{R}^p$ a set of ordered measured temperatures with respect to the top tray. As before, consider the one time-constant models

$$\frac{\Delta x_D}{\Delta R} = \frac{K_{Rx}\exp(-\theta s)}{\tau_{Rx}s+1} \tag{18}$$

and

$$\frac{\Delta T_{m,j}}{\Delta R} = \frac{K_{RT_j}}{\tau_{RT_j}s+1}, \quad j=1,\ldots,p \tag{19}$$

where $\tau_{Rx} > 0$ and $\tau_{RT_j} > 0$ are the dominant time-constants, K_{Rx} and K_{RT_j} are steady-state gains, $\Delta R = R - \overline{R}$, $\Delta x_D = x_D - \bar{x}_D$, $\Delta T_{m,j} = T_{m,j} - \overline{T}_{m,j}$, and $\overline{R}$, $\bar{x}_D$ and $\overline{T}_m$ are, respectively, nominal reflux flow rate, distillate composition and measured tray temperatures. The idea for multiple temperature cascade control is as follows:

- (Step 1) Use $\Delta T_{m,1}$ as a virtual control input to control the distillate composition Δx_D. This provides the set point trajectory $\Delta T_{m,1,\text{ref}}$ for the first slave loop.

Table 1. Case Study Configuration

System	Acetone(1)-Ethanol(2)
Thermodynamic model	UNIQUAC
Number of trays	20 incl. condenser and reboiler
Feed tray	10
Column pressure	101.3 kPa
Feed flow rate	100 kmol/h
Feed composition	0.5/0.5 mole fraction
Nominal distillate flow rate	50 kmol/h
Nominal external reflux ratio (R/D)	2.26
Nominal external boilup ratio (V/B)	2.85
Nominal distillate composition	0.9477
Nominal bottom composition	0.0521
Nominal temperature in tray 4	332.09 K
Nominal temperature in tray 6	332.88 K
Nominal temperature in tray 8	334.01 K
Nominal temperature in tray 13	335.82 K
Nominal temperature in tray 17	336.39 K

• (Step j up to p) Use $\Delta T_{m,j}$ as a virtual control input to control the tray temperature $\Delta T_{m,j-1}$ along the trajectory $\Delta T_{m,j-1,\text{ref}}$. This provides the set point trajectory $\Delta T_{m,j,\text{ref}}$ to the j-th slave loop.

• (Step $p+1$) Use ΔR to control the tray temperature $\Delta T_{m,p}$ along the trajectory $\Delta T_{m,p,\text{ref}}$.

The final product is a cascade controller in the master/p-slave configuration, where the p-slave controller is a series of p temperature controllers ordered in cascade form. Notice that this procedure reduces to the single master/slave controller when $p=1$.

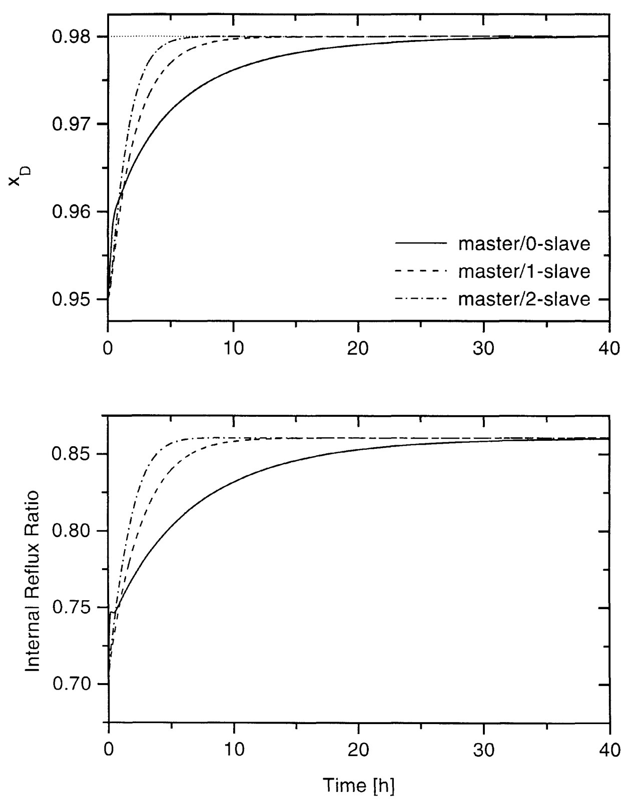

Figure 3. Dynamical behavior of the regulated output and the control input under a step set point change.

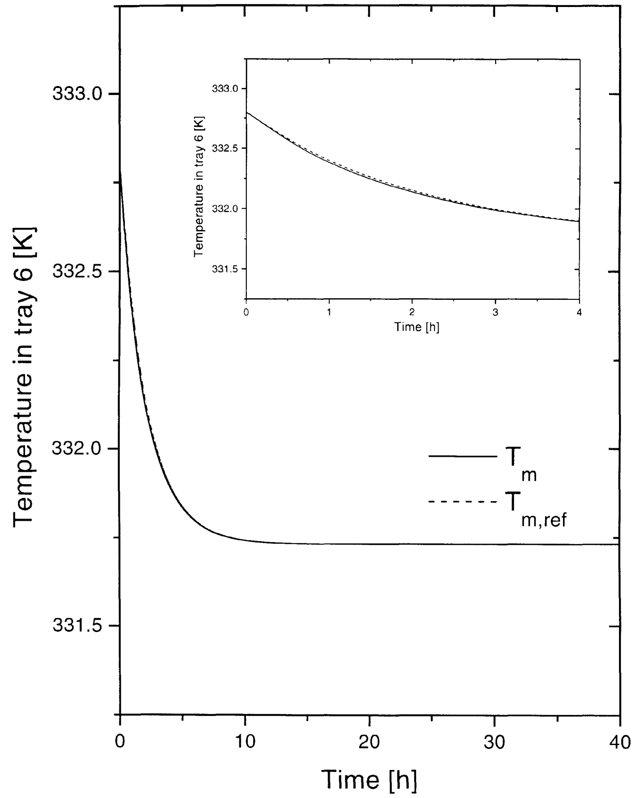

Figure 4. Actual T_m and reference $T_{m,\text{ref}}$ temperatures.

Notice that T_m tracks the output of the master controller $T_{m,\text{ref}}$, which provides continuous protection against external disturbances.

For cascade control design, consider the input-output maps obtained from Eqs. 18 and 19: $\Delta R \overset{G_{T,p}(s)}{\to} \Delta T_{m,p}$, $\Delta T_{m,j} \overset{G_{T,j+1}(s)}{\to} \Delta T_{m,j+1}$, $j=1, \ldots, p-1$, and $\Delta T_{m,1} \overset{G_x(s)}{\to} \Delta x_D$, where

$$G_x(s) \overset{\text{def}}{=} K_{T_1x}\left(\frac{\tau_{RT_1}s+1}{\tau_{Rx}s+1}\right)\exp(-\theta s), \quad K_{T_1x} \overset{\text{def}}{=} K_{Rx}/K_{RT_1}$$

is a stable, nonminimum-phase (due to the time-delay $\exp(-\theta s)$) transfer function

$$G_{T,j}(s) = K_{T_j}\left(\frac{\tau_{RT_j}s+1}{\tau_{RT_{j+1}}s+1}\right), \quad K_{T_j} \overset{\text{def}}{=} K_{RT_j}/K_{RT_{j-1}}$$

is a stable, minimum-phase transfer function, and

$$G_{T,p}(s) = \frac{K_{RT_p}}{\tau_{RT_p}s+1}$$

is also a stable, minimum-phase transfer function. The master controller design is based on the input-output model $G_x(s)$. As described before, a low-gain pure integral controller suffices to yield regulation of the system output about the desired set point value $x_{D,\text{ref}}$. In this way, the proposed master controller can be given as

$$T_{m,1,\text{ref}} = \overline{T}_{m,1} + K_{T_1x}^{-1}\tau_{M,I}^{-1}\int_0^t [x_{D,\text{ref}} - x_D(\sigma)]d\sigma$$

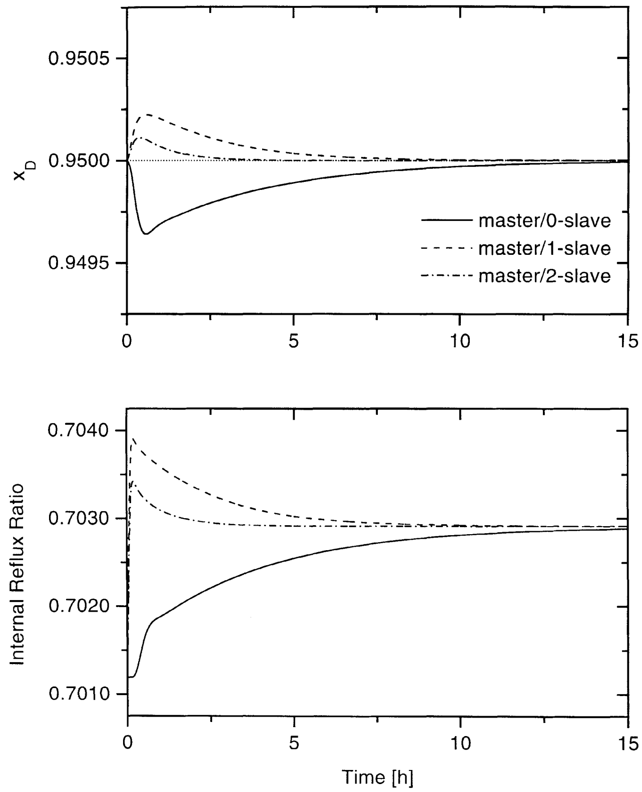

Figure 5. Dynamical behavior of the regulated output and the control input for +10% feed flow rate change.

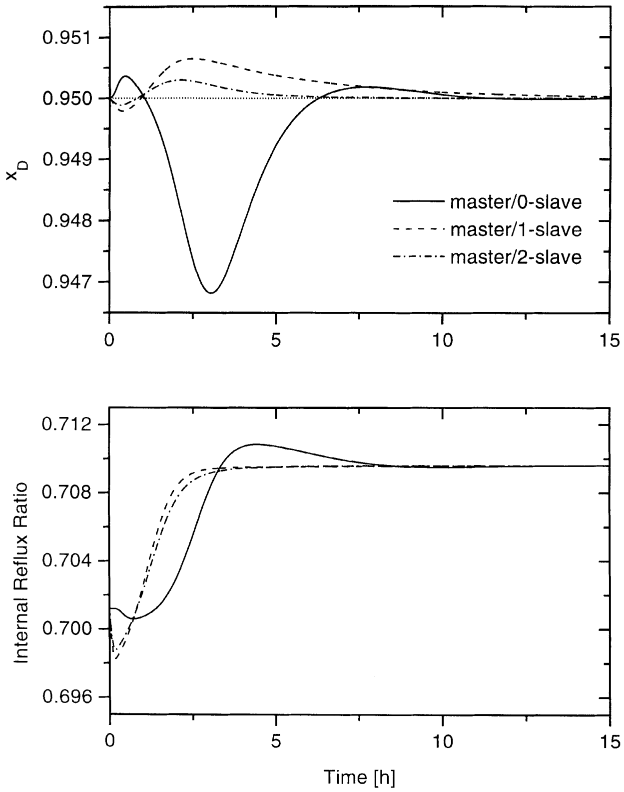

Figure 6. Dynamical behavior of the regulated output and the control input for −10% feed flow rate change.

where $\tau_{M,I} > 0$ is the master integral time-constant. The design of the first $p-1$ temperature slave controllers is based on the models $G_{T,j}(s)$.

Having built the master controller to regulate the distillate composition by manipulating the trajectory $T_{m,1,\text{ref}}$ of the first measured tray temperature, the task for the j-th slave control design is to obtain a controller that manipulates the trajectory $T_{m,j+1,\text{ref}}$ in such a way that the tray temperature tracks the reference signal $T_{m,j,\text{ref}}(t)$ generated by the $(j-1)$-th slave controller. By taking $\eta_j(T_{m,j}(t), T_{m,j+1}(t))$, the modeling error associated to the input-output path $\Delta T_{m,j} \rightarrow \Delta T_{m,j+1}$, the construction of the j-th slave temperature controller is similar to that discussed earlier. All the slave controllers have a linear PI control structure with $T_{m,j} - T_{m,j,\text{ref}}$ being the corresponding j-th tracking error. Besides, the p-slave construction allows the introduction of control saturations at all levels, hence, taking $T_{m,j,\text{ref}}^{\min}$ and $T_{m,j,\text{ref}}^{\max}$, the minimum and maximum j-th reference values, respectively. This nested saturations reduces under-shooting and over-shooting induced by high-gain estimations.

Numerical Simulations

The separation of acetone (*subindex 1*) and ethanol (*subindex 2*) will be considered to illustrate the functioning of the proposed controller. The model and system characteristics are given in Table 1, with tray 6 used to locate the secondary measurements for a master/1-slave configuration, and trays 4 and 8 used to locate the secondary measurements for a master/2-slave configuration (see Tolliver and McCune (1980) for a discussion on optimal temperature control trays). The thermodynamic parameters were taken from Gmehling and Onken (1977).

According to models 1 and 2, the plant parameters are: $K_{Rx} = 0.001265$ h·kmol^{-1}, $K_{RT_4} = -0.03185$ K·h·kmol^{-1}, $K_{RT_6} = -0.04469$ K·h·kmol^{-1}, $K_{RT_8} = -0.05309$ K·h·kmol^{-1}, $\tau_{Rx} = 1.116$ h, $\tau_{RT_4} = 0.682$ h, $\tau_{RT_6} = 0.558$ h, and $\tau_{RT_8} = 0.620$ h (see Figure 2). For acetone-ethanol mixtures, typical measurement times in chromatography equipment is of the order of 0.1 h. We take $\theta = 0.16$ h to account for internal liquid flow delays also. A direct composition PI controller (that is, a master/0-slave controller) tuned with available IMC rules with a closed-loop time constant of $0.2\tau_{Rx} = 0.223$ h (Chien and Fruehauf, 1990) is used for comparison.

Following the tuning guidelines described earlier, the following master and slave control parameters were used: $\tau_{M,I} = 0.2$ h, $\tau_{S,c,j} = 0.3$ h, and $\tau_{S,e,j} = 0.1$ h. The master/slave cascade configuration Eq. 7, and Eq. 13 as described earlier, was used to implement the composition cascade configuration in both the master/1-slave and master/2-slave cascade control configurations.

(a) *Set point changes.* Figure 3 shows the dynamics of the regulated output and the control input when a step set point change is made from 0.95 to 0.98. Notice that the performance of the direct composition control deteriorates rapidly. In fact, it is shown that it takes a much longer time for x_D to

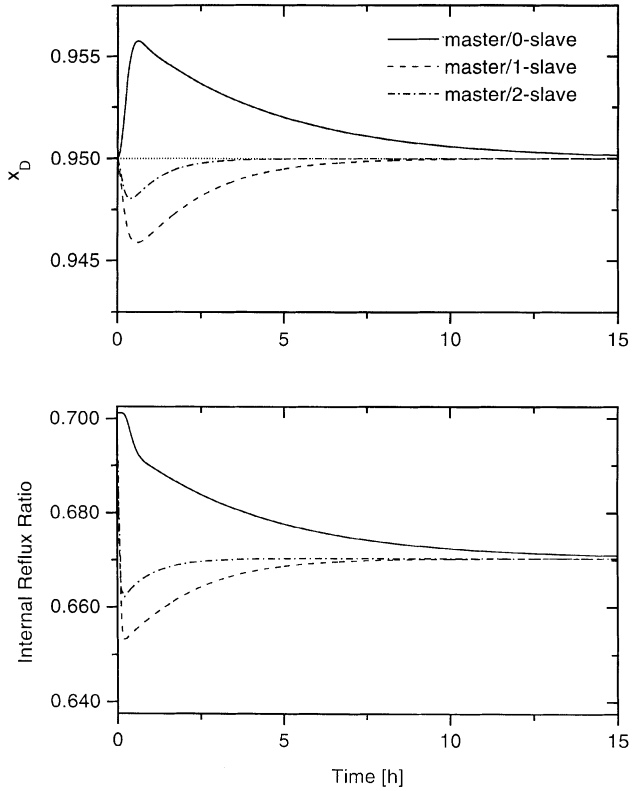

Figure 7. Dynamical behavior of the regulated output and the control input for a feed composition change, from $z_1 = 0.5$ to $z_1 = 0.6$.

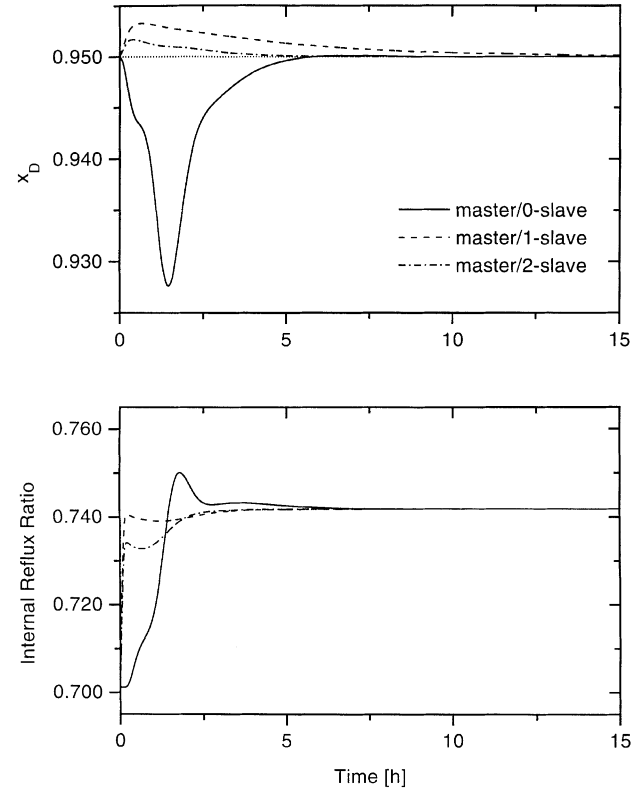

Figure 8. Dynamical behavior of the regulated output and the control input for a feed composition change, from $z_1 = 0.5$ to $z_1 = 0.4$.

return to the set point using the direct composition control. The composition cascade control shows better performance. This is because the slave controller provides fast track of the master control output $T_{m,\mathrm{ref}}$, which yields a better transient response. This fact is illustrated for the master/1-slave configuration in Figure 4. Notice that a slight enhancement of the set point change transient is obtained by the introduction of a second temperature measurement (that is, for the master/2-slave configuration).

(b) *Feed flow rate disturbances*. Figures 5 and 6 show the dynamic response for a step disturbance of $+10\%$ and -10%, respectively, in the feed flow rate. Compared with its noncascade counterpart, the proposed temperature cascade control provides better feedforward action such that the outputs do not deviate too far from the set point. In contrast to the set point change case illustrated by Figure 3, an important enhancement in the controller performance is obtained with the addition of a second temperature measurement. This can be explained from the fact that feed disturbance are quickly detected and damped by the 2-slave controller. On the other hand, direct composition control (that is, master/0-slave configuration) does not have feedforward compensation provided by the temperature loop. Hence, in view of long composition time-delays, direct composition control does not provide protection against external disturbances. Notice that there exists an asymmetric response, but in both cases better disturbance rejections can be achieved using the proposed controller.

(c) *Feed composition disturbances*. Let us look at the response of the controls to a step change in feed composition, which is a severe disturbance. Figures 7 and 8 show the dynamic response for a change of the feed composition from $z_1 = 0.5$ to $z_1 = 0.6$ and from $z_1 = 0.5$ to $z_1 = 0.4$, respectively. As in the previous cases, there exists an asymmetric response. Nevertheless, the proposed cascade control shows a much better performance than the noncascade controller. Again, little overshoot is observed with cascade control.

(d) *Effect of temperature measurement location*. The location of temperature measurements has a major effect on the performance of cascade control (Tolliver and McCune, 1980). Optimal location of temperature measurements is an important structure control problem that deserves a detailed analysis (Wolff and Skogestad, 1996). However, the analysis of this control problem is beyond the scope of this article. To illustrate the effects of location of temperature measurements, Figure 9 presents the dynamics of the regulated distillation composition and the reflux ratio for three different tray temperatures, and a -10% feed flow rate disturbance. For this distillation system, it is noted that the closer the temperature measurement to the top of the column, the better the disturbance rejection. Of course, this fact must not be seen as a conclusion for general distillation systems.

(e) *Measurement noise*. Measurement noise is always present in practical situations. Since the objective of the slave controllers is to track the corresponding temperature trajectory quickly, such control loop can contain a high-gain com-

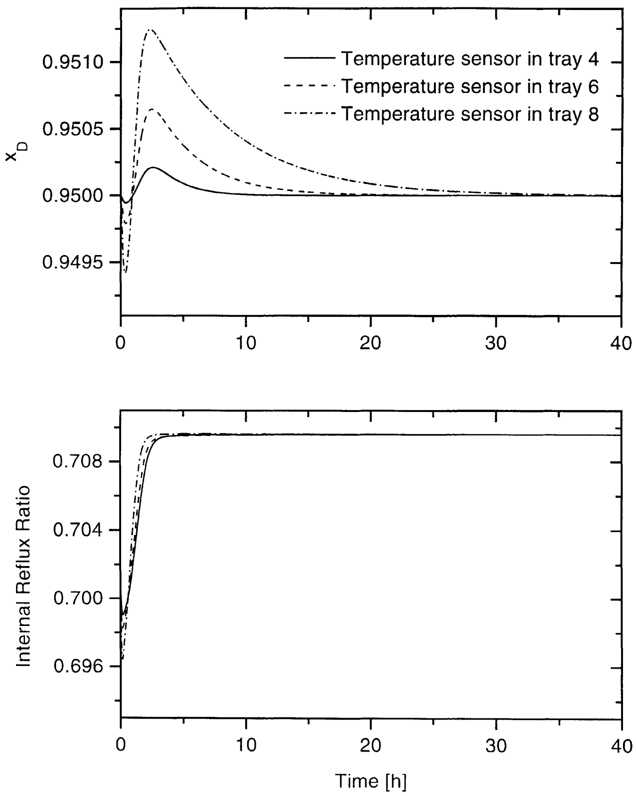

Figure 9. Effects of the location of the temperature measurement on the performance of the proposed cascade control configuration.

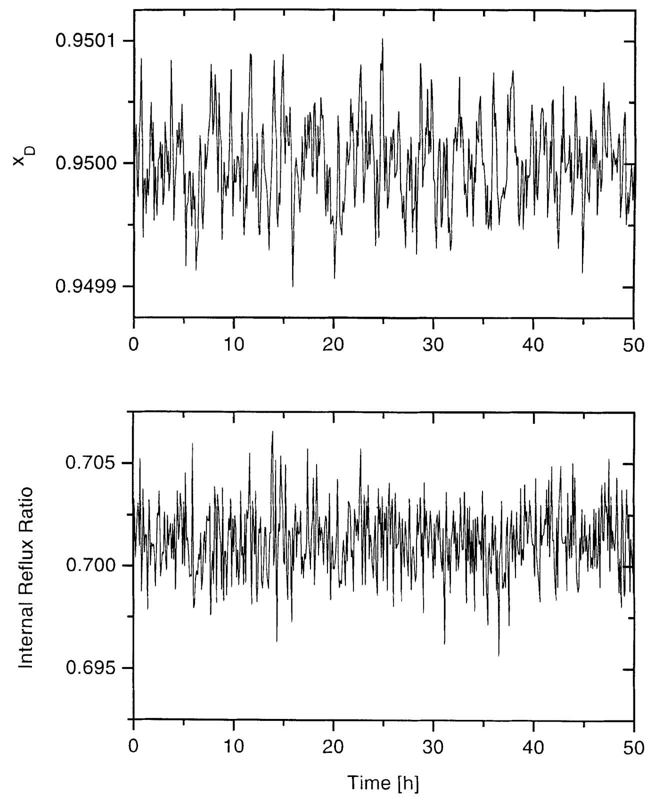

Figure 10. Performance of the cascade controller under measurement noise.

ponent. Figure 10 presents the response of proposed cascade control configuration when the measured composition and temperatures are subjected to ± 0.0001 and ± 2 K measurement noise, respectively. As can be seen in the figure, the cascade controller is able to provide acceptable performance without excessive amplification of measurement noise.

(f) *Comparison with conventional cascade control.* Figure 11 presents the comparison of the proposed master/1-slave cascade control configuration with a conventional cascade control for a -10% feed flow rate disturbance. The conventional cascade control system is constructed as follows: in a first step, a proportional temperature loop is used and tuned to provide a nominal closed-loop time constant of about 0.5 h; while, in a second step, with the temperature loop in automatic, a PI composition loop tuned with IMC tuning rules is used. For a fair comparison, the backstepping-based cascade control configuration is tuned to provide close control gains both in the proportional temperature and PI composition loops (see Eq. 18). This procedure is carried out by adjusting the slave closed-loop and estimation time-constants (see Eq. 19). In this way, the difference in performance would rely basically in the underlying structure of the proposed and conventional cascade control schemes. Figure 11 shows that the backstepping-based cascade controller performs better than the conventional one. This is so because of the backstepping-based cascade controller contains a structure that extracts the performance induced by the master (slow) loop. In fact, the temperature trajectory required by the mas-

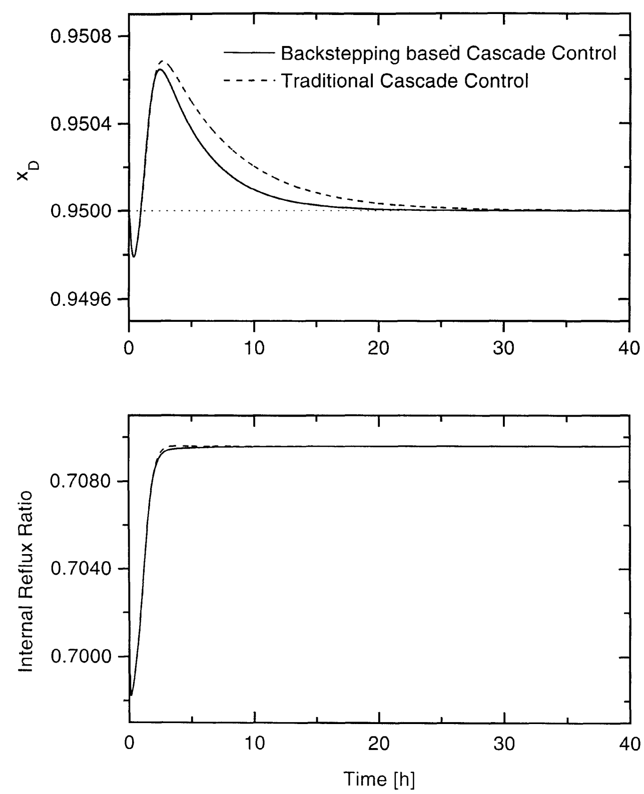

Figure 11. Backstepping-based vs. conventional cascade controllers.

ter loop is quickly tracked by the master loop, hence, providing fast protection against feed disturbances.

(g) *Dual composition control*. As discussed in the sections above, in principle, the backstepping design methodology can be extended to the case of (decentralized) dual composition control. Although care must be taken to ensure stability in the face of ill-conditioned top/bottom loops coupling (Morari and Zafiriou, 1989; Wolff and Skogestad, 1996), we proceed to apply the proposed cascade control methodology in a straightforward way just by taking the diagonal response of two-input two-output plant, where the vapor flow rate V is used in the bottom of the distillation column to regulate the bottom composition. The computed models from a step-response with nominal bottom composition $\bar{x}_B = 0.0521$ and composition time delay $\theta = 0.16$ h are

$$\frac{\Delta x_B}{\Delta V} = \frac{-0.007295\,\exp\,(-0.16s)}{3.54s+1}$$

for the bottom composition, and

$$\frac{\Delta T_{13}}{\Delta V} = \frac{2.0973}{2.585s+1}$$

and

$$\frac{\Delta T_{17}}{\Delta V} = \frac{1.9056}{1.815s+1}$$

for the temperatures in the 13th and the 17th trays. Notice that the stripping dynamics are slower than the rectifying dynamics. Figure 12 shows the response of a double cascade controller when a change of the feed composition from $z_1 = 0.5$ to $z_1 = 0.4$ is made at $t = 0$ h; then, at $t = 40$ h, the feed composition is changed to $z_1 = 0.6$, and a $+10\%$ disturbance is made in the flow rate fed simultaneously with a change of the feed composition to $z_1 = 0.5$ at $t = 80$ h. As can be seen in the figure, there are small deviations from the set points. Similar behaviors as those displayed by the single composition control (see Figure 10) when the measured signals are contaminated with measurement noise.

These numerical simulations show that the composition cascade control is so responsive and the overall control structure is so flexible that the feed to the column can suffer strong disturbances without operator intervention (that is, all controllers remain in automatic). At times, the feed to the column is frequently lowered or even stopped because downstream equipment is plugged. This responsive and flexible

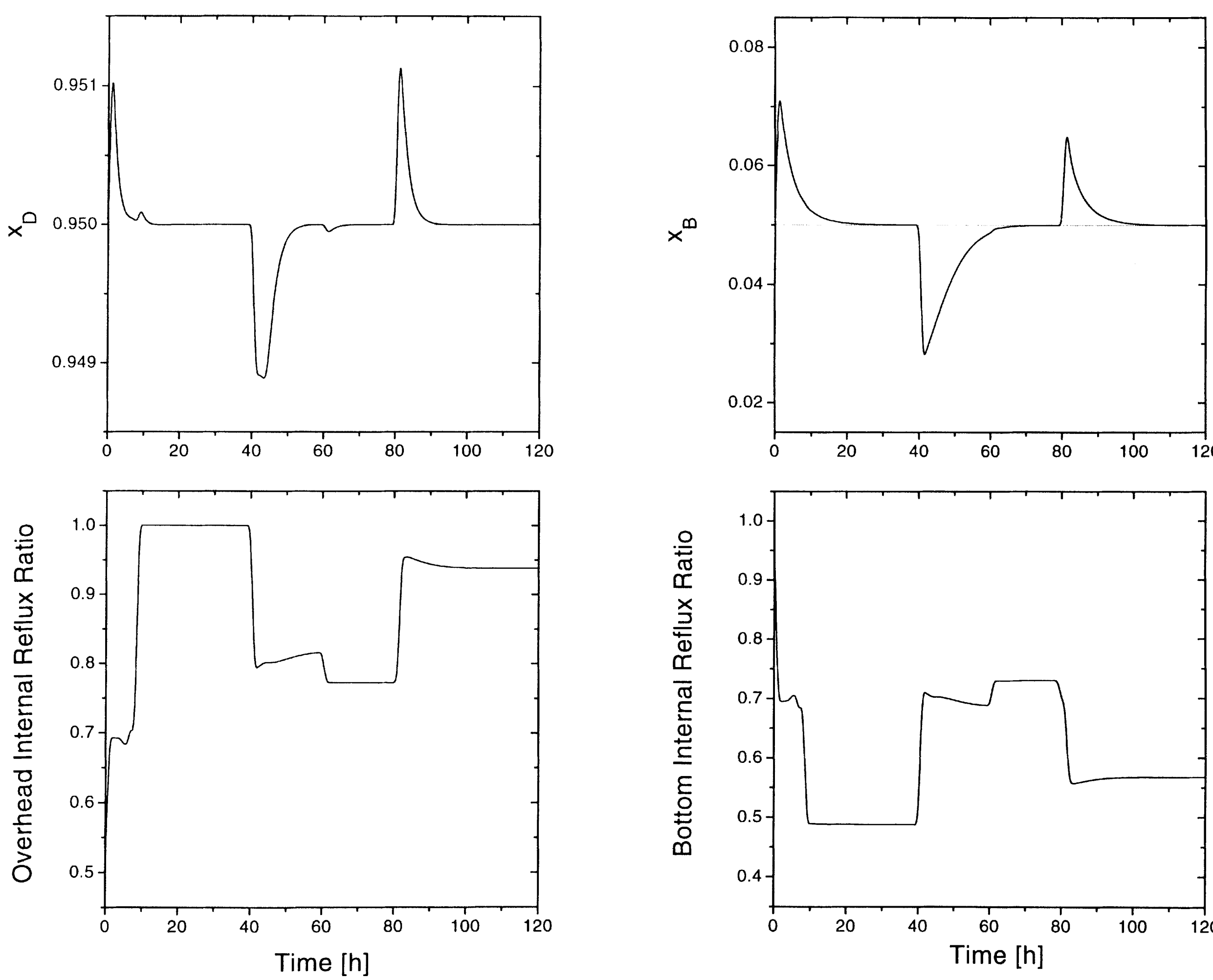

Figure 12. Performance of the cascade control scheme for dual composition control.

control structure allows the column feed to be lowered or stopped and started with no delays.

Conclusions

The systematic construction, tuning, and stability aspects of the composition-temperature cascade control problem of distillation columns has been addressed by employing the backstepping approach associated to the constructive nonlinear control methodology. First, it was shown how the backstepping approach is the natural way to systematically construct the composition-temperature cascade controller, according to the following rationale: exploit the online temperature measurement: (a) to set a virtual manipulated (temperature) input that passivates the controller design by approximating the underlying high-order uncertain (and, possibly, poorly robust) input-(reflux/heat load) output (composition) column model with a set of low-order, simple, linear, and output matched models in cascade interconnection; (b) to obtain a fast estimate of the (persistent) modeling errors in the input-(reflux/heat load) output (temperature) model, enabling the fast compensation of load and model errors in the fast (slave) control loop; and (c) to design an integral-action master composition controller to compensate for set point deviations and input-(temperature) output (composition) modeling errors. The resulting controller constriction resembles the ones reported earlier in cascade control designs, and presents three improvements: (a) the master composition controller drives the slave temperature one with information on the temperature set point and its time-derivative; (b) the input-output matching term of the slave controller is effectively compensated via a simple reduced-order observer that runs faster than the slave loop dynamics; and (c) opens the possibility of enhancing the controller performance by incorporating more than one temperature measurement in each cascade controller. These construction and robust functioning features were: (a) backed-up by a robust convergence condition in conjunction with a closed-loop dynamics characterization; and (b) corroborated by the implementation of the single-point, dual-point, and multiple-temperature control designs to a binary acetone-ethanol column simulated with a detailed model.

Methodology speaking, the present study suggests a way to proceed in the employment of advanced stability-oriented control techniques to address or revisit application-oriented problems in chemical process control.

Acknowledgments

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Appendix: Proof of Theorem 1

Before providing the proof of Theorem 1, we need to prove some preliminary results. Let $\mathcal{L}_2$ and $\mathcal{L}_\infty$ be, respectively, the set of square-integrable and uniformly bounded signals. The next lemma states that, if the slave controller provides uniformly bounded and vanishing tracking error $\xi_m(t)$ (that is, $\xi_m \in \mathcal{L}_2 \cap \mathcal{L}_\infty$), then the distillation composition converges to the desired set point $x_{D,\mathrm{ref}}$.

Lemma A.1

Let $\xi_m = T_m - T_{m,\mathrm{ref}} = \Delta T_m - \Delta T_{m,\mathrm{ref}}$ be the slave tracking error. If $\xi_m \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, for sufficiently large master integral time-constant $\tau_{M,I} > 0$, $x_D(t) \to x_{D,\mathrm{ref}}$ as $t \to \infty$.

Proof

Consider the input-output path $\Delta T_m \to \Delta x_D$ given by Eq. 4. Take ΔT_m as the virtual control input $\Delta T_{m,\mathrm{ref}}$

$$\frac{\Delta x_D}{\Delta T_{m,\mathrm{ref}}} = K_{Tx}\left(\frac{\tau_{RT}s+1}{\tau_{Rx}s+1}\right)\exp\left(-\theta s\right)$$

From the Small Gain Theorem (Desoer and Vidyasagar, 1975; Bonnet et al., 2000), we know that the master controller $\Delta T_{m,\text{ref}}(s) = C_M(s)e_M(s)$, where $e_M(s) \overset{\text{def}}{=} x_{D,\text{ref}} - x_D$ and $C_M(s) = K_{M,I}s^{-1}$, $K_{M,I} = K_{Tx}^{-1}\tau_{M,I}^{-1}$ ensures asymptotic regulation of the distillation composition x_D about the desired set point $x_{D,\text{ref}}$, for sufficiently large values of the integral time-constant $\tau_{M,I}$. This is, the master loop is stable for low integral gains $K_{M,I}$. However, the actual master "control input" is ΔT_m, rather than $\Delta T_{m,\text{ref}}$. If $\xi_m = T_m - T_{m,\text{ref}}$ is the slave tracking error, we can write the actual plant (Eq. 4) as

$$\frac{\Delta x_D}{\Delta T_{m,\text{ref}} - \xi_m} = K_{Tx}\left(\frac{\tau_{RT}s+1}{\tau_{Rx}s+1}\right)\exp(-\theta s)$$

So, we can see the tracking error ξ_m as an additive disturbance acting in the input channel $\Delta T_{m,\text{ref}}$. For simplicity in notation, let

$$G_M(s) \overset{\text{def}}{=} K_{Tx}\left(\frac{\tau_{RT}s+1}{\tau_{Rx}s+1}\right)\exp(-\theta s)$$

Under the master control $C_M(s)$, the closed-loop system is

$$\Delta x_D = H_1(s)\Delta x_{D,\text{ref}} + H_2(s)\xi_m$$

where

$$H_1(s) = G_M(s)C_M(s)/[1 + G_M(s)C_M(s)]$$
$$H_2(s) = -G_M(s)/[1 + G_M(s)C_M(s)]$$

Since the input-output path $\Delta x_{D,\text{ref}} \to \Delta x_D$ is asymptotically stable for sufficiently large values of $\tau_{M,I} > 0$, $H_1(s)$ and $H_2(s)$ are asymptotically stable transfer functions. Consequently, if $\xi_m \in \mathfrak{L}_2 \cap \mathfrak{L}_\infty$, we conclude that $\Delta x_D \to \Delta x_{D,\text{ref}}$, and, hence, $x_D \to x_{D,\text{ref}}$ (see Desoer and Vidyasagar, 1975). ■

Consequently, we should prove that the proposed slave controller gives uniformly bounded and asymptotically vanishing tracking error. In the next proposition, the uniform boundedness and convergence of the tracking error ξ_m is guaranteed.

Proposition A.2

The exact slave controller (Eq. 10) yields $\xi_m \in \mathfrak{L}_2 \cap \mathfrak{L}_\infty$.

Proof

It is a straightforward consequence of the construction of the exact slave controller. In fact, the corresponding slave closed-loop is $\dot{\xi}_m = -\tau_{S,c}^{-1}\xi_m$, which is linear and stable. Hence, $\xi_m \in \mathfrak{L}_2 \cap \mathfrak{L}_\infty$. ■

This result states that the exact slave controller (Eq. 10) solves the cascade control design problem; namely, it guarantees $x_D \to x_{D,\text{ref}}$ asymptotically. However, an implementation of Eq. 10 is made via the approximation given by Eq. 13.

Proposition A.3

Assume that $\tilde{K}_{RT}^{-1}(\partial\eta/\partial R) + 1 > 0$. There exists a positive constant $\tau_{S,e}^{\max}$ such that, for all $0 < \tau_{S,e} < \tau_{S,e}^{\max}$, the approximate slave controller (Eq. 13) guarantees $\xi_m \in \mathfrak{L}_2 \cap \mathfrak{L}_\infty$. That is, the composition regulation error is uniformly bounded and converges asymptotically to zero as long as $0 < \tau_{S,e} < \tau_{S,e}^{\max}$.

Proof

Let

$$\begin{aligned} \dot{w}_M &= A_M w_M + B_M\,\mathfrak{D}_\theta(T_m - \overline{T}_m) \\ x_D &= C_M w_M \end{aligned} \tag{A1}$$

be a state-space realization of the system (Eq. 4), where $\mathfrak{D}_\theta(.)$ is the time-delay operator (that is, $\mathfrak{D}_\theta(x(t)) = x(t-\theta)$). This state-space realization exists since the system (Eq. 4) is causal. System (A1) can be rewritten as

$$\begin{aligned} \dot{w}_M &= A_M w_M + B_M\,\mathfrak{D}_\theta(T_{m,\text{ref}} - \overline{T}_m) + B_M\,\mathfrak{D}_\theta(\xi_m) \\ x_D &= C_M w_M \end{aligned} \tag{A2}$$

where, as defined before, $\xi_m = T_m - T_{m,\text{ref}}$ is the tracking error. The system (Eq. A2) under the master integral control is

$$\begin{aligned} \dot{w}_M &= A_M w_M + B_M\,\mathfrak{D}_\theta(w_{M,I}) + B_M\,\mathfrak{D}_\theta(\xi_m) \\ \dot{w}_{M,I} &= K_{M,I}(x_{D,\text{ref}} - C_M w_M) \\ x_D &= C_M w_M \end{aligned} \tag{A3}$$

Let $(\overline{w}_M^T, \overline{w}_{M,I})^T$ be the equilibrium point of system (Eq. A3) with $\xi_m = 0$. Introduce the deviation states $z_M \overset{\text{def}}{=} w_M - \overline{w}_M$ and $z_{M,I} = w_{M,I} - \overline{w}_{M,I}$. In this way, we get

$$\begin{aligned} \dot{z}_M &= A_M z_M + B_M\,\mathfrak{D}_\theta(z_{M,I}) + B_M\,\mathfrak{D}_\theta(\xi_m) \\ \dot{z}_{M,I} &= -K_{M,I}C_M\bar{z}_{M,I} \end{aligned} \tag{A4}$$

On the other hand, let $\epsilon_\eta \overset{\text{def}}{=} \eta - \hat{\eta}$ be the estimation error. Under the slave feedback (Eq. 13), we get

$$\dot{\xi}_m = -\tau_{c,S}^{-1}\xi_m + \epsilon_\eta \tag{A5}$$

In this way, the controlled process is composed by the master (Eq. A4) and slave (Eq. A5) dynamics. By virtue of Lemma A.2, for a sufficiently large master integral time-constant $\tau_{M,I} > 0$ and $\epsilon_\eta = 0$, the system (Eq. A4), (Eq. A5) is asymptotically stable about the origin. Now, let us compute the dynamics of the estimation error ϵ_η. From Eq. 11 and the fact that $\eta = \dot{T}_m - \tau_{RT}^{-1}(-T_m + \overline{T}_m) - \tilde{K}_{RT}^{-1}(R - \overline{R})$, we get

$$\dot{\epsilon}_\eta = -\tau_{e,S}^{-1}\epsilon_\eta + \dot{\eta} \tag{A6}$$

To compute the time-derivative $\dot{\eta}$, we recall that the modeling error is a function of both the input and the output signals, that is, $\eta(t) = \eta[T_m(t),\ R(t)]$. Consequently,

$$\dot{\eta} = (\partial\eta/\partial T_m)\dot{T}_m + (\partial\eta/\partial R)\dot{R} \tag{A7}$$

In $(z_M,\ z_{M,I},\ \xi_m)$-coordinates we can write $\dot{T}_m = f_m(z_M,\ z_{M,I},\ \xi_m,\ \epsilon_\eta)$, where $f_m(z_M,\ z_{M,I},\ \xi_m)$ is a linear function of its arguments with $f_m(0,\ 0,\ 0) = 0$. On the other hand, since by assumption $R = R^c$, we can use Eq. 10 to get $\dot{R} = -\tau_{S,e}^{-1}\tilde{K}_{RT}^{-1}\epsilon_\eta + \phi(z_M,\ z_{M,I},\ \xi_m,\ \epsilon_\eta)$, where $\phi(z_M,\ z_{M,I},\ \xi_m,\ \epsilon_\eta)$ is a linear function of its arguments with $\phi(0,\ 0,\ 0,\ 0) = 0$. Hence, from Eqs. A6 and A7, we get

$$\dot{\epsilon}_\eta = -\tau_{S,e}^{-1}\kappa(t)\epsilon_\eta + \Phi(z_M,\ z_{M,I},\ \xi_m,\ \epsilon_\eta) \tag{A8}$$

where

$$\Phi(z_M,\ z_{M,I},\ \xi_m,\ \epsilon_\eta) = (\partial\eta/\partial T_m)f_m(z_M,\ z_{M,I},\ \xi_m,\ \epsilon_\eta) + (\partial\eta/\partial R)\phi(z_M,\ z_{M,I},\ \xi_m,\ \epsilon_\eta) \tag{A9}$$

and

$$\kappa(t) = \tilde{K}_{RT}^{-1}(\partial\eta/\partial R) + 1 \tag{A10}$$

By assumption $1 + \tilde{K}_{TR}^{-1}(\partial\eta/\partial R) > 0$. Hence, the system $\dot{\epsilon}_\eta = -\tau_{S,e}^{-1}\kappa(t)\epsilon_\eta$ is exponentially stable for all $\tau_{S,e} > 0$. For $\tau_{S,e}^{-1}$ sufficiently small, such closed-loop dynamics are in the form of a standard singularly perturbed system. In this way, the result follows from a direct application of Hoppensteadt's Theorem 3 (Hoppensteadt, 1974). ■

Proof of Theorem 1

(a) It is a direct consequence of the design of composition cascade controller and Proposition A.3; (b) This limit can easily obtained from a matched expansion in terms of the singular perturbation parameter $\tau_{S,c}$ (see Hoppensteadt's Theorem 3).

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